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Note

A Numerical Solution for the Laplace Equation with Normal Derivative Boundary Condition

INTRODUCTION

A method is presented for the numerical finite difference solution of the Laplace equation with Neumann conditions around the whole boundary. The method is based on reducing the Laplace equation to a set of time-dependent partial differential equations, using the values of the normal derivatives on the boundaries and solving the time-dependent equations until a steady-state solution is reasonably obtained.

The Laplace equation

$$\Delta \Phi = 0 \tag{1}$$

is generally associated with equilibrium or steady-state problems and is the governing equation in a large number of problems. The velocity potential for the steady flow of incompressible non-viscous fluid satisfies Laplace's equation and is the mathematical way of expressing the idea that the rate at which such fluid enters any given region is equal to the rate at which it leaves it. In the application of the vector and scalar potential method to the numerical solution of two-and three-dimensional Navier–Stokes equations by Aregbesola and Burley [1] the importance of the solution of this equation is mentioned.

In a simple region the solution may be obtained analytically. This is not always possible in regions of irregular shapes and numerical solution is a likely approach. When the values of the dependent variable Φ are specified throughout the boundaries the numerical conditions several computational difficulties are presented as discussed in Aregsbesola and Burley [1]. In this case the Laplace equation has a solution if and only if

$$\int_V \nabla^2 \Phi \, dV = \int_V \nabla \cdot \nabla \Phi \, dV = \int_S \nabla \Phi \cdot \mathbf{dS} = \int_S \frac{\partial \Phi}{\partial n} \, dS = 0, \tag{2}$$

where S denotes the surface enclosing the volume V, and a solution is obtained to within an arbitrary constant. In most cases the gradients of the dependent variable Φ only are required so that the arbitrary constant is irrelevant. The numerical approach must therefore conform with the condition in Eq. (2).

The usual numerical method of solving Laplace's equation is to find the finite difference approximation to Eq. (1) and that of the associated boundary conditions

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and solve them iteratively using possibly the successive over-relation iterative method (SOR). With this approach probable sources of error are in the finite difference representation of the normal derivatives on the boundaries and at corners, especially where the normal derivatives may not be continuous. The errors introduced are likely to effect the condition in (2) adversely.

Some of the computational difficulties in the numerical solution of Laplace's equation with normal boundary conditions therefore are ensuring that the boundary conditions are satisfied at each stage of the computation so that the condition in Eq. (2) is always satisfied and ensuring that a solution to within an arbitrary constant is actually obtained. The approach discussed in this work uses the values of the normal derivatives and not the difference approximation of the derivatives. The idea is to solve Laplace's equation $\Delta\Phi = 0$ by solving the time-dependent problem

$$R\frac{\partial\Phi}{\partial t} + \frac{\partial^2\Phi}{\partial t^2} = \Delta\Phi$$

through a system of equations that do not require differencing of the normal derivatives of Φ at the boundary.

A similar approach is used by Yanenko [2] to solve Laplace's equation through a system of equations equivalent to that discussed here. These equations are then solved by the method of fractional steps. The resulting equations in this work are solved using a finite difference scheme iteration method and an attempt is made to obtain a close estimate of the optimum value of R that gives a quick convergence. The method is tested for solving a potential flow problem in a two-dimensional rectangular region.

Finite Difference Approximation

The Laplace equation (1) can be reduced to a set of time-dependent partial differential equations if variables U, V, W are introduced such that

$$\frac{\partial\Phi}{\partial x} = RU + \frac{\partial U}{\partial t}, \tag{3}$$

$$\frac{\partial\Phi}{\partial y} = RV + \frac{\partial V}{\partial t}, \tag{4}$$

$$\frac{\partial\Phi}{\partial z} = RW + \frac{\partial W}{\partial t} \tag{5}$$

and these equations are substituted into the equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \frac{\partial\Phi}{\partial t}, \tag{6}$$

the result being

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = R \frac{\partial \Phi}{\partial t} + \frac{\partial^2 \Phi}{\partial t^2}, \tag{7}$$

where R is a positive constant. The parameter R is introduced to play a role analogous to that of the relaxation parameter in the SOR method. As a steady solution is approached, $\partial\Phi/\partial t$, $\partial^2\Phi/\partial t^2$ tend to zero, so that Eq. (7) becomes identical to Eq. (1). The boundary conditions imposed on the variables U, V, W are such that they are independent of time and since the normal derivatives are specified at these points, the boundary values of these variables are obtained from the relations

$$\frac{\partial \Phi}{\partial x} = RU, \qquad \frac{\partial \Phi}{\partial y} = RV, \qquad \frac{\partial \Phi}{\partial z} = RW, \tag{8}$$

which are the boundary conditions on Eqs. (3) to (5).

To improve the accuracy of the computation an interlaced mesh system at illustrated in [1] is recommended. The variable Φ is computed at the centre point surrounded by points where variables U, V, W are iterated. The finite difference approximation of Eq. (1) and the accompanying boundary conditions are straightforward and can be found in any standard text. The finite differencing of Eq. (3) will be considered; those of Eqs. (4) and (5) are similar. Centred at time n, Eq. (3) reduces to

$$\begin{aligned}(\Phi(I+1,J,K) - \Phi(I,J,K))_n/\Delta x \\ = 0.5R(U(I,J,K)_{n+1/2} + U(I,J,K)_{n-1/2}) \\ + (U(I,J,K)_{n+1/2} - U(I,J,K)_{n-1/2})/\Delta t,\end{aligned} \tag{9}$$

where Δt is the time increment and Δx is the space increment in the direction along x-axis. Similarly the finite difference approximation of Eq. (6) is

$$\begin{aligned}(U(I,J,K) - U(I-1,J,K))/\Delta x + (V(I,J,K) - V(I,J-1,K))/\Delta y \\ + (W(I,J,K) - W(I,J,K-1))/\Delta z]_n \\ = (\Phi(I,J,K)_{n+1} - \Phi(I,J,K)_n)/\Delta t,\end{aligned} \tag{10}$$

where n is the nth time iteration. On the boundaries the finite differencing of the normal derivatives is not necessary; only values of the variables U, V, W are needed for the computation.

Rough Estimate of the Optimum Value of the Parameter R

Consider the Laplace equation in one dimension,

$$\frac{\partial^2 \Phi}{\partial x^2} = 0, \qquad \text{for} \quad 0 < x < 2, \tag{11}$$

with

$$\frac{\partial \Phi}{\partial x} = 1, \qquad \text{at} \quad x = 0, 2. \tag{12}$$

The solution of Eq. (11) with condition (12) is $\Phi = x$, for $0 \leqslant x \leqslant 2$, plus an arbitrary constant. These equations can be replaced by the set of equations

$$\frac{\partial \Phi}{\partial x} = UR + \frac{\partial U}{\partial t}, \tag{13}$$

$$\frac{\partial U}{\partial x} = \frac{\partial \Phi}{\partial t} \qquad \text{for} \quad 0 < x < 2, \tag{14}$$

and

$$\frac{\partial \Phi}{\partial x} = UR = 1 \qquad \text{at} \quad x = 0, 2. \tag{15}$$

If $\Phi_{I,n}$ denotes the value of the nth time iteration of Φ at point I, the finite difference approximation of Eqs. (13) and (14) are respectively

$$(\Phi_{I,n+1} - \Phi_{I-1,n+1})/h = 0.5R(U_{I,n+1} + U_{I,n}) + (U_{I,n+1} - U_{I,n})/k, \tag{16}$$

$$(U_{I+1,n} - U_{I,n})/h = (\Phi_{I,n+1} - \Phi_{I,n})/k, \tag{17}$$

where $h = \Delta x$ and $k = \Delta t$.

By applying the finite difference scheme (17) and then (16) and continuing in that order, Eq. (16) can be written in the form

$$U_{I,n+1} = \rho U_{I,n} + \beta Y_{I,n+1} + \gamma Z_{I,n},$$

where

$$\rho = (2 - 2Rk)/(2 + Rk),$$

$$\gamma = k\beta/h = 2k^2/h^2(2 + Rk),$$

$$Y_{I,n} = \Phi_{I+1,n} - \Phi_{I,n} = h\left[\frac{\partial \Phi}{\partial x}\right]_{I,n},$$

and

$$\begin{aligned} Z_{I,n} &= U_{I+1,n-1} - 2U_{I,n-1} + U_{I-1,n-1} \\ &\quad + U_{I+1,n} - 2U_{I,n} + U_{I-1,n} \\ &= h^2\left[\frac{\partial^2 U}{\partial x^2}\right]_{I,n-1} + h^2\left[\frac{\partial^2 U}{\partial x^2}\right]_{I,n}. \end{aligned} \tag{18}$$

Thus

$$U_{I,n+1} = \rho U_{I,n} + \beta h \left[\frac{\partial \Phi}{\partial x}\right]_{I,n+1} + \gamma h^2 \left(\left[\frac{\partial^2 U}{\partial x^2}\right]_{I,n+1} + \left[\frac{\partial^2 U}{\partial x^2}\right]_{I,n} \right). \tag{19}$$

Also from Eq. (17) we obtain

$$\Phi_{I,n+1} - \Phi_{I-1,n+1} = \Phi_{I,n} - \Phi_{I-1,n} + k(U_{I+1,n} - 2U_{I,n} + U_{I-1,n})/h$$

giving

$$\left[\frac{\partial \Phi}{\partial x}\right]_{I,n+1} = \left[\frac{\partial \Phi}{\partial x}\right]_{I,n} + k\left[\frac{\partial^2 U}{\partial x^2}\right]_{I,n}. \tag{20}$$

$(\partial\Phi/\partial x)_{I,0}$ and $(\partial^2 U/\partial x^2)_{I,0}$ exist, since they are computed from the initial values of $\Phi_{I,0}$ and $U_{I,0}$. As a result from Eqs. (20) and (19), $(\partial\Phi/\partial x)_{I,1}$ and then $U_{I,1}$ exist. Similarly $(\partial^2 U/\partial x^2)_{I,1}$, $(\partial\Phi/\partial x)_{I,2}$ and $U_{I,2}$ exist since $U_{I,1}$ exists, and so on. It follows that $(\partial^2 U/\partial x^2)_{I,n}$, $(\partial\Phi/\partial x)_{I,n+1}$ and $U_{I,n+1}$ exist.

Since Φ is differentiable, $\lim_{n\to\infty}(\partial\Phi/\partial x)_{I,n} = (\partial\Phi/\partial x)_I$ and so from Eq. (20),

$$\lim_{n\to\infty}\left[\frac{\partial^2 U}{\partial x^2}\right]_{I,n} = \frac{1}{k}\lim_{n\to\infty}\left(\left[\frac{\partial \Phi}{\partial x}\right]_{I,n+1} - \left[\frac{\partial \Phi}{\partial x}\right]_{I,n}\right) = 0.$$

Thus from Eq. (19)

$$\lim_{n\to\infty}(U_{I,n+1} - \rho U_{I,n})$$

$$= \lim_{n\to\infty}\left(\beta h\left[\frac{\partial \Phi}{\partial x}\right]_{I,n+1} + \gamma h^2\left(\left[\frac{\partial^2 U}{\partial x^2}\right]_{I,n-1} + \left[\frac{\partial^2 U}{\partial x^2}\right]_{I,n}\right)\right),$$

giving

$$U_I = \frac{\beta h}{1-\rho}\left[\frac{\partial \Phi}{\partial x}\right]_I = \frac{\gamma}{1-\rho}\left[\frac{h^2}{k}\frac{\partial \Phi}{\partial x}\right]_I. \tag{21}$$

If

$$M = \sup_{0\leqslant k\to\infty}\left[|U_{I,k}|, \left|\frac{h^2}{k}\frac{\partial \Phi}{\partial x}\right|\right],$$

then

$$|U_I| = M\gamma/(1-\rho). \tag{22}$$

Equating $\gamma/(1-\rho)$ to 1 reduces the variation between M and $U_{I,n}$ to a minimum during the iterations. Thus a value ρ_0 of ρ corresponding to $1 = \gamma/(1-\rho_0)$ is very close to the optimum value of ρ that gives the fastest rate of convergence. Thus

$$\rho_0 = 1 - \gamma, \tag{23}$$

giving

$$(2 - R_0 k)/(2 + R_0 k) = 1 - 2k^2/h^2(2 + R_0 k).$$

Hence

$$R_0 = k/h_2. \tag{24}$$

This result tested by solving Eqs. (16) and (17) with $h = 0.1$ and $k = 0.025$ for various values of the parameter R. The results show that $R = k/h^2 = 2.5$, in this problem, is the optimum value of in the one-dimensional problem.

When two or three space coordinates are involved a similar analysis can be carried out as follows. Consider Eqs. (6) and (3) to (5). These equations can be written in the forms

$$\Phi_{I,n+1} = \Phi_{I,n} + k(\nabla \cdot \mathbf{V})_{I,n}$$

and

$$U_{I,n+1} = \rho^n U_{I,n} + \gamma_1 A_{I,n},$$
$$V_{J,n+1} = \rho^n V_{J,n} + \gamma_2 B_{J,n},$$
$$W_{K,n+1} = \rho^n W_{K,n} + \gamma_3 C_{K,n},$$

where ρ is as defined before,

$\mathbf{V} = (U, V, W)$,

$$\Delta x = h, \qquad \Delta y = d, \qquad \Delta z = m,$$
$$A_{I,n} = h[\Phi_{I+1,n+1} - \Phi_{I,n+1} + (\nabla \cdot \mathbf{V})_{I+1,n} - (\nabla \cdot \mathbf{V})_{I,n}],$$
$$B_{I,n} = d[\Phi_{J+1,n+1} - \Phi_{J,n+1} + (\nabla \cdot \mathbf{V})_{J+1,n} - (\nabla \cdot \mathbf{V})_{J,n}],$$
$$C_{K,n} = m[\Phi_{K+1,n+1} - \Phi_{K,n+1} + (\nabla \cdot \mathbf{V})_{K+1,n} - (\nabla \cdot \mathbf{V})_{K,n}],$$

and

$$\gamma_1 = 2k^2/h^2(2 + Rk),$$
$$\gamma_2 = 2k^2/d^2(2 + Rk),$$
$$\gamma_3 = 2k^2/m^2(2 + Rk).$$

Following an argument similar to that used, before, it can be shown that

$$1 \leqslant \gamma_1/(1 - \rho), \qquad 1 \leqslant \gamma_2/(1 - \rho), \qquad 1 \leqslant \gamma_3/(1 - \rho).$$

Thus

$$1 \leqslant \min(\gamma_1, \gamma_2, \gamma_3)/(1 - \rho).$$

By equating $\min(\gamma_1, \gamma_2, \gamma_3)/(1-\rho)$ to 1, an estimate of the optimum value of R can be obtained as

$$R_0 = \min(k/h^2, k/d^2, k/m^2). \tag{25}$$

Given Δx, Δy and Δz, the stability and rate of convergence depend on Δt. An increase from zero in the value of Δt improves the rate of convergence up to certain stage but a further increase in the value may lead to instability. However, it was not possible to obtain a mathematical expression for the stability condition based on Δt. For the test problem stability is possible for $0 < \Delta t^2/\Delta x^2 < 0.5$.

Application to a Two-Dimensional Potential Flow

The method is applied to solve a potential flow in a rectangular region in which

$$\nabla^2 \Phi = 0, \qquad \text{for} \quad 0 < x < 2, \quad 0 < y < 2, \tag{26}$$

subject to the boundary conditions

$$\frac{\partial \Phi}{\partial y} = 0, \qquad \text{for} \quad 0 < x < 2, \quad y = 0,$$

$$0 < x < 1.5, \quad y = 2; \tag{27}$$

$$\frac{\partial \Phi}{\partial y} = 1, \qquad \text{for} \quad 1.5 < x < 2, \quad y = 2; \tag{28}$$

$$\frac{\partial \Phi}{\partial x} = 0, \qquad \text{for} \quad 0.5 < y < 2, \quad x = 0; \tag{29}$$

$$\frac{\partial \Phi}{\partial x} = 1, \qquad \text{for} \quad 0 < y < 0.5, \quad x = 0. \tag{30}$$

The whole region is subdivided into 20×20 meshes with 21×21 mesh points such that $\Delta x = \Delta y = 0.1$. For each problem and in each circle of iteration maximum variation of Φ, $\max|\Phi_{n+1} - \Phi_n|$, between two successive iterations in the whole region is computed. Various values of Δt ranging from 1/80 to 1/15 with the values of the parameter R ranging from 1 to 5 were used for the computation. For a particular value of Δt it was found that the optimum value of R that gives the fastest rate of convergence is $R = \Delta t/\Delta x^2$ as estimated earlier.

The results obtained by the SOR method and the method discussed here show that the variation using the former method tends to zero faster than that of the latter. The values of $\max|\Phi_{n+1} - \Phi_n|$ when $n = 40$, 120, 300 for $R = 5$ and $\Delta t = 0.05$ are 3.479×10^{-3}, 3.95×10^{-4} and 2.7×10^{-6}; those from the SOR method are 2.626×10^{-3}, 3.95×10^{-5} and 6.00×10^{-7} respectively. However, from the computation of the values of $\max|\nabla^2 \Phi|$, it was found that the method discussed here

performs better. For $n = 50$, 100, 200, 400, the values of $\max |\nabla^2 \Phi|$ are 0.2793, 0.0687, 0.00431, 0.000023 and 0.000011; the values from the SOR method are 0.2055, 0.0123, 0.00048, 0.00033 and 0.00033, respectively. It may be pointed out that the ultimate goal of the numerical solution is to get a solution that makes $\max |\nabla^2 \Phi| = 0$.

While the variation of $\max |\Phi_{n+1} - \Phi_n|$ tends to zero more quickly using the SOR method, with the method discussed here, the $\max |\nabla^2 \Phi|$ values are smaller than the corresponding values obtained from the SOR method when n is large.

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